Chapter 6

Continuous Random Variables

We previously examined several different probability distributions for discrete random variables, in particular the binomial, Poisson, and negative binomial distributions. These distribution are suitable for modeling observations that are counts of some type, such as the number of plants in a quadrat or the number of females vs. males in a sample. Many variables in biology are continuous, however, such as the length and weight of organisms, quantities associated with populations such as birth, mortality, and growth rates, and chemical concentrations. We will now examine continuous random variables and their associated distributions that are used to model these quantities, in particular the **uniform and normal distributions**. The uniform distribution is often used to generate random sampling points in one- and two-dimensional areas. For example, we could use the uniform distribution to select a random point along a transect to sample, or a random x, y coordinate within a field to place a sampling quadrat. It also a useful starting point for understanding continuous distributions because of its simplicity. We then turn to the normal distribution, which forms the basis of many statistical procedures. Many biological variables have a distribution close to normal, or if initially non-normal can often be transformed to more closely resemble the normal distribution.

Discrete random variables have a function $f(y)$ that directly provides the probabilities for events that are integers, such as $Y = 0$, $Y = 3$, and so forth (see Chapter 5). However, events for continuous random variables are in the form of intervals. For example, we will be interested in finding the probability for events like $1 < Y < 3$ or $Y > 5$. Continuous random variables use a different kind of function, called a **probability density function**, to find the probabilities for events. For an event like $1 < Y < 3$, probabilities are found by integrating the probability density function (finding the area under the function) over this interval. This process will be explained in more detail below. For many continuous random variables, such as the normal distribution, there exist tables of these integrals and probabilities for certain useful intervals. Note that events like $Y = 3$ have zero probability for continuous random variables, because this implies an interval of zero width and so the integral is zero. This makes some intuitive sense, because it is unlikely that a continuous quantity Y would take a value exactly equal to 3 to many decimal places.

6.1 Uniform distribution

Suppose that we have two constants, a and b, with $a < b$. A random variable Y has a uniform distribution if an observation is equally likely to occur anywhere between a and b, but never occurs outside this interval. The probability density for the uniform distribution is defined by the equation

$$f(y) = \frac{1}{b - a} \tag{6.1}$$

for $a \leq y \leq b$ (Mood et al. 1974). Outside of this interval, we have $f(y) = 0$. The quantities a and b are the parameters of the uniform distribution. The uniform distribution for $a = 0$, $b = 1$ is shown below (Fig. 6.1). The uniform distribution gets its name from the fact that its density is uniform over the interval a to b.

Note that the density simply describes a square with a length and width of one, implying an area equal to one. This is an important property of probability density functions in general – the area under $f(y)$ is always equal to one. Also shown is the uniform density for $a = 0$ and $b = 2$ (Fig. 6.2). It is lower but wider than the previous example, and also has an area of one.

Figure 6.1: Uniform probability density for $a = 0, b = 1$

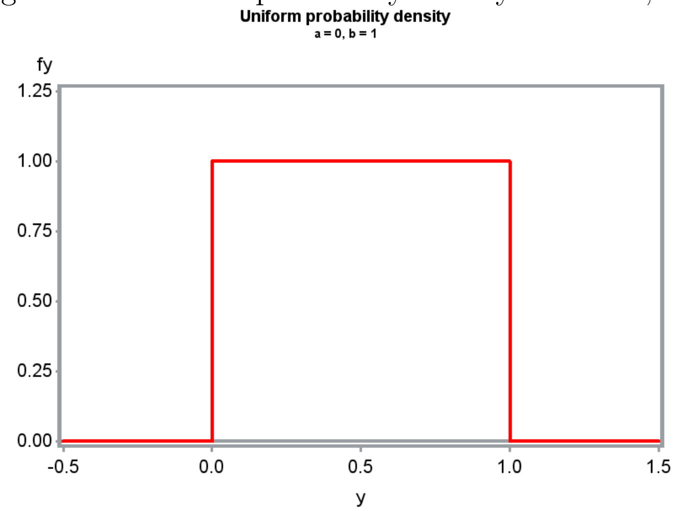
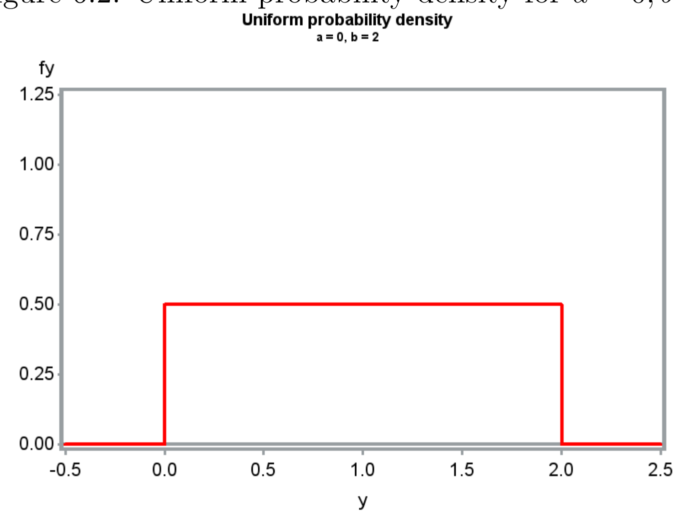


Figure 6.2: Uniform probability density for $a = 0, b = 2$



Probabilities for the uniform distribution are calculated by finding the area under the probability density function. This is relatively easy to do because of the simple form of the probability density. Suppose Y is a uniform random variable, and $a = 0$ and $b = 1$. What is the probability that an observed Y lies within the interval 0.5 to 0.75? We have

$$P[0.5 < Y < 0.75] = \int_{0.5}^{0.75} \frac{1}{b-a} dy \tag{6.2}$$

$$= \int_{0.5}^{0.75} \frac{1}{1-0} dy = y|_{0.5}^{0.75} \tag{6.3}$$

$$= 0.75 - 0.5 = 0.25. \tag{6.4}$$

We could also have found this probability without any calculus. It is just the area under $f(y)$ between 0.5 and 0.75, calculated as length $\times$ height $= (0.75 - 0.5) \times 1 = 0.25$.

Here are two more examples. Suppose that for $a = 0$ and $b = 2$, we want to find the probability that $0.2 < Y < 0.4$. The height of the density function in this case is $1/(b-a) = 1/(2-0) = 0.5$. We therefore have $P[0.2 < Y < 0.4] = (0.4 - 0.2) \times 0.5 = 0.1$. Now suppose we want the probability that $0 < Y < 2$. We have $P[0 < Y < 2] = (2-0) \times 0.5 = 1$. This also follows from the fact that $f(y)$ is a probability density function which has an area of one, and the interval $0 < Y < 2$ encompasses the entire range of $f(y)$.

The **cumulative distribution function** for a continuous random variable is defined as the quantity

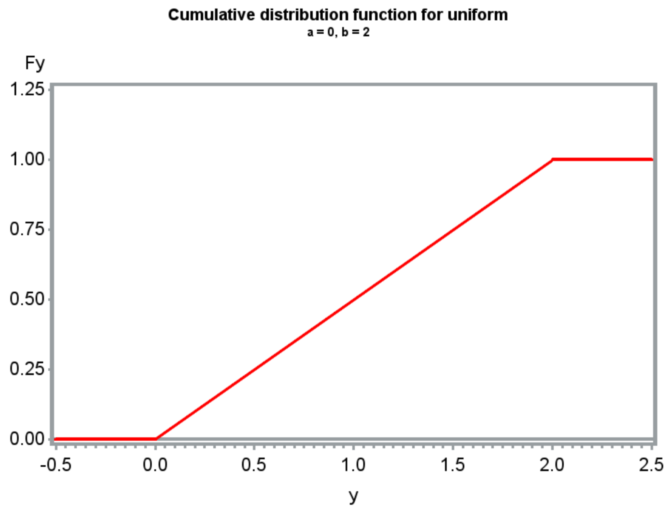
$$F(y) = P[Y < y] = \int_{-\infty}^{y} f(z)dz. \tag{6.5}$$

This function is just the probability to the left of y. The function $F(y)$ increases from 0 to 1 as y increases. If we carry out this integral for the uniform distribution, we get the function

$$F(y) = \frac{y-a}{b-a} \tag{6.6}$$

for $a \leq y \leq b$. In addition, $F(y) = 0$ for $y < a$, and $F(y) = 1$ for $y > b$. Figure 6.3 shows the cumulative distribution function for the uniform distribution corresponding to Fig. 6.2. Note that it increases linearly between

Figure 6.3: Cumulative distribution function for the uniform distribution, with $a = 0, b = 2$



a and b, as the probability to the left of y accumulates. The cumulative distribution function has many uses in statistics, especially for continuous random variables.

The uniform distribution has a number of common applications. It is possible to generate a stream of random numbers that have uniform distribution using software, which can then be used to generate random observations for other distributions, including discrete distributions as well as the normal distribution. The uniform distribution can also be used to generate random sampling points along a transect for ecological studies, or random x, y coordinates for placing quadrats within an area (see below). It can also be used to generate random samples from a population, or randomly order treatments in an experiment.

6.1.1 Random sampling coordinates - SAS demo

A common application of the uniform distribution is to generate random sampling coordinates. SAS can generate random observations with a uniform distribution using the function `ranuni`. For this function, the parameter values of the uniform distribution are set at $a = 0$ and $b = 1$.

However, we will often want observations for other parameter values, especially other values of b. It can be shown that if Y has a uniform distribution with $a = 0$ and $b = 1$, then the variable $Y' = cY$ has a uniform distribution with $a = 0$ and $b = c$, where c is any positive number. This fact enables us to generate uniform random variables with any value of b.

For example, suppose we want to generate random sampling coordinates along a 100 m transect using the uniform distribution. If Y has a uniform distribution with $a = 0$ and $b = 1$, then $Y' = 100Y$ has a uniform distribution with $a = 0$ and $b = 100$. Values of Y generated in this fashion will give us sampling coordinates uniformly distributed between 0 and 100 m.

We will illustrate this process using a SAS program to generate random sampling coordinates for a 100 m transect and also a 200 × 100 m rectangular area. A call to `gplot` is used to plot the random coordinates. See SAS program and output below.

SAS Program

```
* randcoords.sas;
options pageno=1 linesize=80;
goptions reset=all;
title "Generate random sampling coordinates";
* Generate n random coordinates along a c m transect;
data transect;
        * Sample size n;
        n = 20;
        * Multiplying by c gives a uniform random variable with a=0, b=c;
        c = 100;
        do i = 1 to n;
                x = c*ranuni(0);
                output;
        end;
        drop i;
run;
* Print coordinates;
proc print data=transect;
run;
* Generate n random coordinates within a 200 x 100 m area;
data coords;
        * Sample size n;
        n = 200;
        * Multiplying by c_x gives a uniform random variable with a=0, b=c_x;
        c_x = 200;
        * Multiplying by c_y gives a uniform random variable with a=0, b=c_y;
        c_y = 100;
```

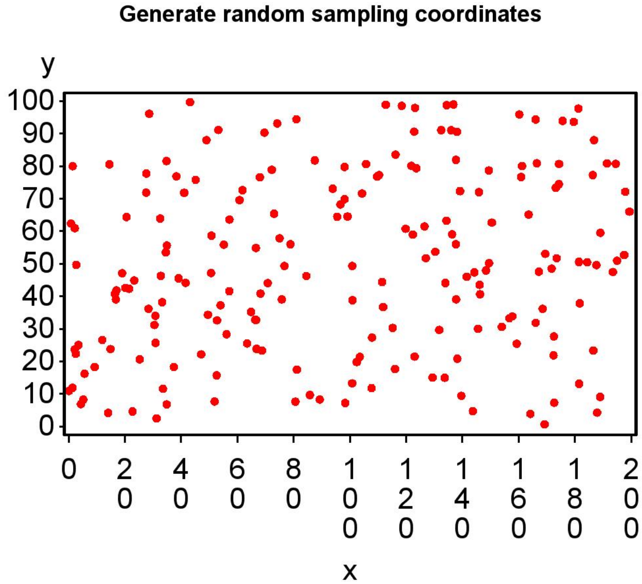
```
        do i = 1 to n;
                x = c_x*ranuni(0);
                y = c_y*ranuni(0);
                output;
        end;
        drop i;
run;
* Print first 25 coordinates;
proc print data=coords(obs=25);
run;
* Show coordinates as a scatterplot;
proc gplot data=coords;
        plot y*x / vaxis=axis1 haxis=axis2;
        symbol1 v=dot c=red;
        axis1 order=(0 to 100 by 10) label=(height=2) value=(height=2)
        width=3 major=(width=2) minor=none;
    axis2 order=(0 to 200 by 20) label=(height=2) value=(height=2)
        width=3 major=(width=2) minor=none;
run;
quit;
```

SAS Output

```
                      Generate random sampling coordinates                   1
                                                    15:53 Monday, April 19, 2010

                             Obs      c         x

                               1     100    19.9499
                               2     100    76.3413
                               3     100    79.9041
                               4     100    15.7759
                               5     100    15.2421
                               6     100    71.3867
                               7     100    23.3531
                               8     100    73.9213
                               9     100    75.5294
                              10     100    55.6698
                              11     100    42.3700
                              12     100    67.0161
                              13     100    23.0314
                              14     100    17.1588
                              15     100    68.1973
                              16     100    20.1917
                              17     100    91.6066
                              18     100    50.2973
```

```
            19    100    84.9498
            20    100    36.2745


     Generate random sampling coordinates                  2
                                 15:53 Monday, April 19, 2010

   Obs    c_x    c_y       x          y

     1    200    100    154.862    21.3515
     2    200    100    160.414    70.8713
     3    200    100    118.344    57.3555
     4    200    100    154.958     4.8716
     5    200    100    173.834    80.7355
     6    200    100     40.852     1.9296
     7    200    100    116.088    94.5155
     8    200    100     13.003     5.9704
     9    200    100     58.785    96.1373
    10    200    100    190.694    18.8834
    11    200    100    180.953    29.0750
    12    200    100    100.127    42.8300
    13    200    100     75.700    47.8597
    14    200    100    127.454    59.8772
    15    200    100     27.703    35.4066
    16    200    100     16.360     7.5101
    17    200    100     43.722    18.8987
    18    200    100    177.311    55.2469
    19    200    100     41.933     2.2553
    20    200    100    101.261    39.6063
    21    200    100    146.369    48.9749
    22    200    100     44.071    96.6252
    23    200    100    146.298    88.8055
    24    200    100    158.129    43.9857
    25    200    100     58.123    66.6462
```

Figure 6.4: Random y, x coordinates for 200×100 m area



6.2 Normal distribution

The normal distribution plays an important role in statistics, with good reason. Biological variables often have a distribution that can be approximated by the normal or can be transformed to be normal. The normal distribution is thus a valid choice for modeling many variables encountered in practice. Many statistical quantities will also have a distribution approaching the normal for large sample sizes. For example, the distribution of the sample mean $\bar{Y}$ will approach the normal distribution as the sample size n increases, thanks to the central limit theorem (see Chapter 7). So, even if the underlying data are non-normal, statistics like $\bar{Y}$ will be normally-distributed for sufficiently large n.

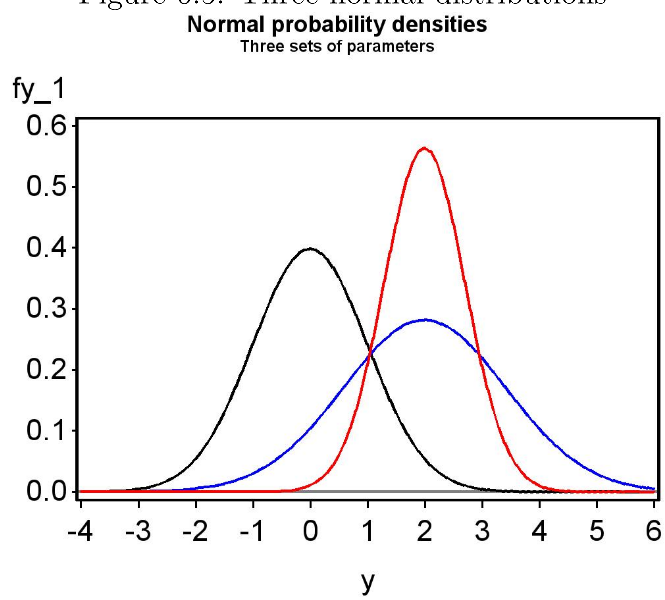
The probability density for the normal distribution is defined by the function

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \tag{6.7}$$

for $\infty < \mu < \infty$ and $\sigma^2 > 0$ (Mood et al. 1974). The normal distribution has two parameters, μ and σ^2. The parameter μ is the mean of the distribution and basically controls its location, while σ^2 is its variance and determines its dispersion or spread. A random variable Y with a normal distribution is often written as $Y \sim N(\mu, \sigma^2)$, where the symbol '$\sim$' stands for 'is distributed as' while 'N' signifies the normal. A random variable with a **standard normal distribution** assumes that $\mu = 0$ and $\sigma^2 = 1$, or $Y \sim N(0, 1)$. The symbol Z is often used to denote a standard normal random variable.

Figure 6.5 shows the bell-shaped normal distribution for three different sets of μ and σ^2 values, and illustrates how these parameters affect its location and shape. As μ is increased the distribution shifts to the right, while an increase in σ^2 causes the distribution to spread out.

Figure 6.5: Three normal distributions



6.2.1 Normal distribution - SAS demo

The SAS program used to generate Fig. 6.5 is listed below. Three different sets of μ and σ^2 values are given in the `data` step of the program (feel free to experiment with other values). The different curves are specified in the `plot` statement for `proc gplot`. The `overlay` option is used to generate a single graph with all three curves, each with different colors specified by the `symbol` statements.

SAS Program

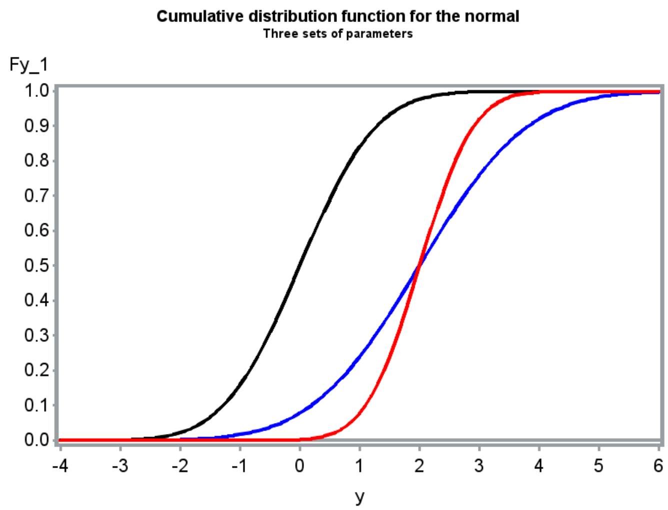
```
* normal_plot3.sas;
options pageno=1 linesize=80;
goptions reset=all;
title "Normal probability densities";
title2 "Three sets of parameters";
data normal_plot;
    * Three sets of normal parameters here;
    mu_1 = 0; sig2_1 = 1;
    mu_2 = 2; sig2_2 = 2;
    mu_3 = 2; sig2_3 = 0.5;
    * Minimum and maximum values of y;
    ymin = -4;
    ymax = 6;
    * Divisions between ymin and ymax (more = smoother graph);
    ydiv = 100;
    * Calculate step length;
    ylength = (ymax-ymin)/ydiv;
    * Find y and f(y) values for the plot;
    do i=0 to ydiv;
        y = ymin + i*ylength;
        * normal probability density function;
        fy_1 = (1/sqrt(2*3.14159*sig2_1))*exp(-((y-mu_1)**2)/(2*sig2_1));
        fy_2 = (1/sqrt(2*3.14159*sig2_2))*exp(-((y-mu_2)**2)/(2*sig2_2));
        fy_3 = (1/sqrt(2*3.14159*sig2_3))*exp(-((y-mu_3)**2)/(2*sig2_3));
        * Output y and fy1, fy2, fy3 to SAS data file;
        output;
    end;
run;
* Print data;
proc print data=normal_plot;
run;
* Plot probability density function;
proc gplot data=normal_plot;
    plot fy_1*y=1 fy_2*y=2 fy_3*y=3 / vref=0 wvref=3 vaxis=axis1 haxis=axis1 overlay;
    symbol1 i=join v=none c=black width=3;
    symbol2 i=join v=none c=blue width=3;
    symbol3 i=join v=none c=red width=3;
    axis1 label=(height=2) value=(height=2) width=3 major=(width=2) minor=none;
run;
quit;
```

The cumulative distribution function for the normal distribution is defined as the quantity

$$F(y) = P[Y < y] = \int_{-\infty}^{y} f(z)dz = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz. \tag{6.8}$$

The values of this integral have to be numerically calculated. Fig. 6.6 shows the cumulative distribution functions for the three normal distributions shown in Fig. 6.5. Note that the mean and variance for the different normal distributions affect the overall location and shape of $F(y)$.

Figure 6.6: Cumulative distribution function for three normal distributions



Like other continuous random variables, events for the normal distribution are in the form of intervals. We can calculate the probabilities for events by finding the area under the normal density function corresponding to the interval. This process is more difficult than for the uniform distribution because $f(y)$ has a more complex shape. However, there exist tables of the area under $f(y)$ for certain intervals that can be used for this purpose, as well as the SAS function `probnorm`. Table Z gives the probabilities for intervals of the form $Z < z$, where Z has a standard normal distribution and $z \geq 0$ (see Chapter 22). The first two digits of z are specified in the left-most column of Table Z, while the third digit is the top row. The values within the table correspond to the probability that $Z < z$, or $P[Z < z]$, i.e., the cumulative distribution function for the standard normal.

6.2.2 Sample calculations - standard normal distribution

We illustrate how Table Z is used to calculate the probabilities for various events listed below. The general strategy is to sketch the interval on the standard normal bell curve, and deduce from this picture how to obtain the probability using Table Z.

1. Find the probability that $Z < 0.55$, or $P[Z < 0.55]$. From Table Z, we see that $P[Z < 0.55] = 0.7088$. See Fig. 6.7 for an illustration of this probability.

2. Find the probability that $0.40 < Z < 1.96$. In this case, the interval is not the same as shown in Table Z, and additional calculations are required. We first find the probabilities for the intervals $Z < 1.96$ and $Z < 0.4$ using Table Z. The probability for $0.40 < Z < 1.96$ should then be the difference between these two probabilities (see Fig. 6.8). We have $P[Z < 1.96] = 0.9750$ and $P[Z < 0.40] = 0.6554$ from Table Z, so $P[0.40 < Z < 1.96] = P[Z < 1.96] - P[Z < 0.40] = 0.9750 - 0.6554 = 0.3196$.

3. Find the probability that $Z > 0.55$. We will use the complement rule to obtain this probability (see Chapter 4). For any event A, we have $P[A^c] = 1 - P[A]$. If A is the event $Z < 0.55$, then A^C corresponds to $Z > 0.55$. Therefore, $P[Z > 0.55] = 1 - P[Z < 0.55] = 1 - 0.7088 = 0.2912$. See also Fig. 6.9.

4. Find the probability that $Z < -1.23$. This problem makes use of the symmetry of the standard normal distribution around zero, as well as the complement rule. By symmetry, we have $P[Z < -1.23] = P[Z > 1.23]$. The complement of $Z < 1.23$ is $Z > 1.23$, and so $P[Z > 1.23] = 1 - P[Z < 1.23] = 1 - 0.8907 = 0.1093$. See Fig. 6.10.

5. Find the probability that $-0.44 < Z < 2.15$. This problem can also be handled using symmetry and the complement rule. We first have

$P[Z < 2.15] = 0.9842$ using Table Z (Fig. 6.11). We then have $P[Z < -0.44] = P[Z > 0.44] = 1 - P[Z < 0.44] = 1 - 0.6700 = 0.3300$ by symmetry (Fig. 6.12). Therefore, $P[-0.44 < Z < 2.15] = P[Z < 2.15] - P[Z < -0.44] = 0.9842 - 0.3300 = 0.6542$.

6. Find a number z_0 such that $P[Z < z_0] = 0.95$. This problem is the inverse of the previous ones. Here, we want to find a value z_0 that gives a certain probability, rather than z_0 being a given quantity and determining the probability. To find z_0, we scan Table Z until we find a value that gives a probability close 0.95. We see that $z_0 = 1.64$ or 1.65 give approximately the right probability.

Figure 6.7: Sample calculation 1

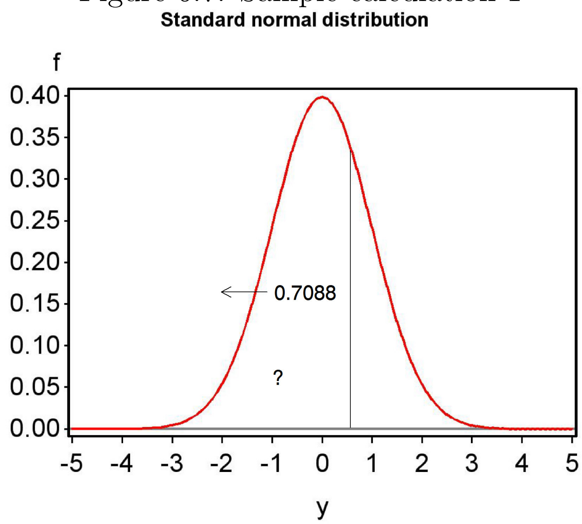


Figure 6.8: Sample calculation 2

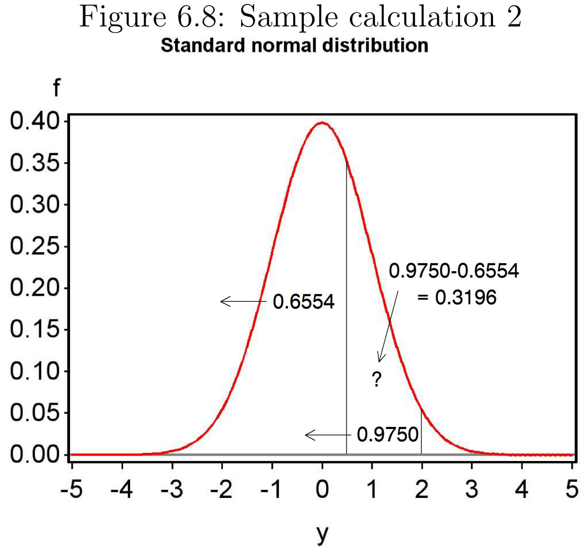


Figure 6.9: Sample calculation 3

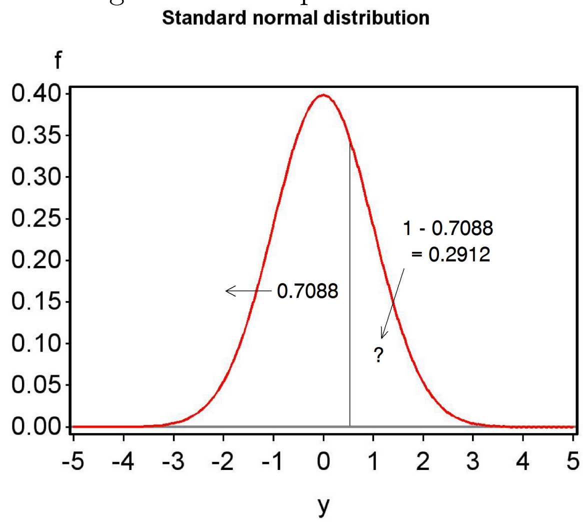


Figure 6.10: Sample calculation 4

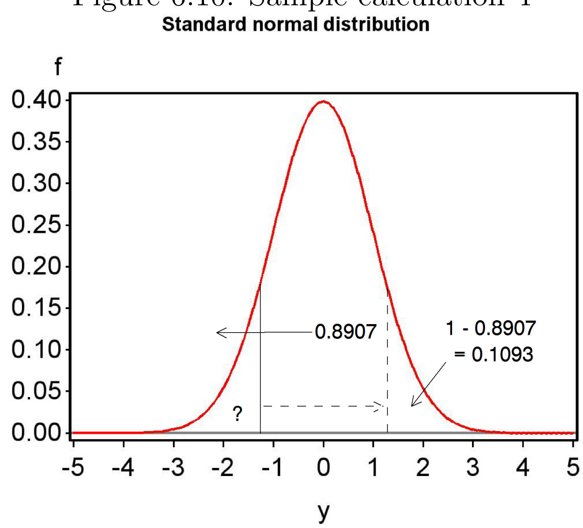


Figure 6.11: Sample calculation 5 - part 1

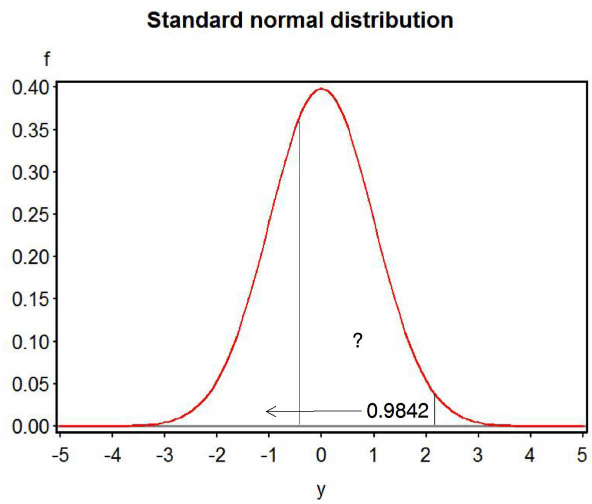
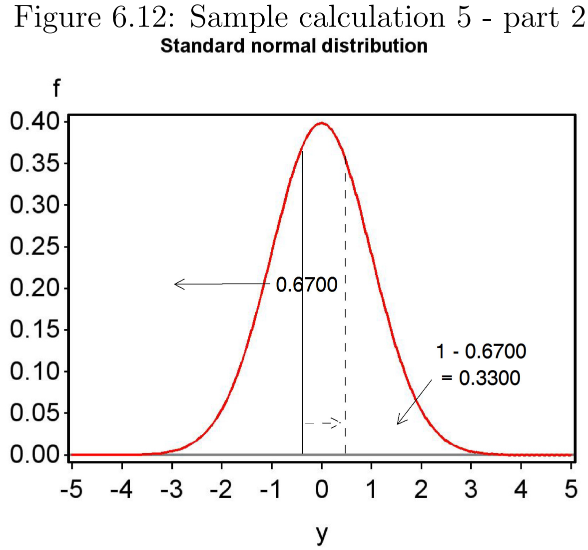


Figure 6.12: Sample calculation 5 - part 2



6.2.3 Sample calculations - other normal distributions

We now examine how probabilities can be calculated for normal distributions that are not standard normal. If $Y \sim N(\mu, \sigma^2)$, it can be shown that the quantity

$$Z = \frac{Y - \mu}{\sigma} \sim N(0, 1) \tag{6.9}$$

Thus, a random variable Y with a normal distribution having any μ or σ^2 can be transformed to a standard normal Z. The transformation works by first centering the random variable Y around zero by subtracting μ, and then dividing by σ so that it has a standard deviation and variance of one. Once Y is transformed to a standard normal Z, we can find probabilities for any event involving Y using Table Z. This process is illustrated below in several sample calculations.

1. Suppose that $Y \sim N(50, 16)$. Find the probability that $Y < 55$. First, we find $\sigma = \sqrt{\sigma^2} = \sqrt{16} = 4$. Using the above equation, we then have

$$\begin{aligned} P[Y < 55] &= P\left[Y - \mu < 55 - \mu\right] &(6.10)\\ &= P\left[\frac{Y - \mu}{\sigma} < \frac{55 - \mu}{\sigma}\right] &(6.11)\\ &= P\left[Z < \frac{55 - 50}{4}\right] &(6.12)\\ &= P[Z < 1.25]. &(6.13)\end{aligned}$$

   We then use Table Z to find that $P[Z < 1.25] = 0.8944$, and so $P[Y < 55] = 0.8944$.

2. Find the probability that $52 < Y < 56$, assuming $Y \sim N(50, 16)$. To find this probability, we first convert the problem to one involving Z. We have

$$\begin{aligned} P[52 < Y < 56] &= P\left[52 - \mu < Y - \mu < 56 - \mu\right] &(6.14)\\ &= P\left[\frac{52 - \mu}{\sigma} < \frac{Y - \mu}{\sigma} < \frac{56 - \mu}{\sigma}\right] &(6.15)\\ &= P\left[\frac{52 - 50}{4} < Z < \frac{56 - 50}{4}\right] &(6.16)\\ &= P[0.50 < Z < 1.50]. &(6.17)\end{aligned}$$

We next find the probabilities for the intervals $Z < 1.50$ and $Z < 0.50$ using Table Z, and then substract them to obtain $P[0.50 < Z < 1.50]$. We have $P[Z < 1.50] = 0.9332$ and $P[Z < 0.50] = 0.6915$, so $P[0.50 < Z < 1.50] = 0.9332 - 0.6915 = 0.2417$. Thus, $P[52 < Y < 56] = 0.2417$.

3. Find the probability that $Y > 54$. We have

$$
\begin{aligned}
P[Y > 54] &= P\left[Y - \mu > 54 - \mu\right] &(6.18)\\
&= P\left[\frac{Y-\mu}{\sigma} > \frac{54-\mu}{\sigma}\right] &(6.19)\\
&= P\left[Z > \frac{54-50}{4}\right] &(6.20)\\
&= P[Z > 1.00]. &(6.21)
\end{aligned}
$$

We next use the complement rule to obtain this probability. We have $P[Z > 1.00] = 1 - P[Z < 1.00] = 1 - 0.8413 = 0.1587$, so $P[Y > 54] = 0.1587$.

4. Find the probability that $Y < 46.5$. We have

$$
\begin{aligned}
P[Y < 46.5] &= P\left[Y - \mu < 46.5 - \mu\right] &(6.22)\\
&= P\left[\frac{Y-\mu}{\sigma} < \frac{46.5-\mu}{\sigma}\right] &(6.23)\\
&= P\left[Z < \frac{46.5-50}{4}\right] &(6.24)\\
&= P[Z < -0.88]. &(6.25)
\end{aligned}
$$

By symmetry, we have $P[Z < -0.88] = P[Z > 0.88]$. The complement of $Z < 0.88$ is $Z > 0.88$, and so $P[Z > 0.88] = 1 - P[Z < 0.88] = 1 - 0.8106 = 0.1093$. So, $P[Y < 46.5] = 0.1093$.

5. Find the probability that $46 < Z < 52$. We have

$$
\begin{aligned}
P[46 < Y < 52] &= P\left[46 - \mu < Y - \mu < 52 - \mu\right] &(6.26)\\
&= P\left[\frac{46-\mu}{\sigma} < \frac{Y-\mu}{\sigma} < \frac{52-\mu}{\sigma}\right] &(6.27)\\
&= P\left[\frac{46-50}{4} < Z < \frac{52-50}{4}\right] &(6.28)\\
&= P[-1.00 < Z < 0.50]. &(6.29)
\end{aligned}
$$

We then use symmetry and the complement rule to find this probability involving Z. We first have $P[Z < 0.50] = 0.6915$ using Table Z. We then have $P[Z < -1.00] = P[Z > 1.00] = 1 - P[Z < 1.00] = 1 - 0.8413 = 0.1587$ by symmetry. Therefore, $P[-1.00 < Z < 0.50] = P[Z < 0.50] - P[Z < -1.00] = 0.6915 - 0.1587 = 0.5328$, and so $P[46 < Y < 52] = 0.5328$.

6. Find a number y_0 such that $P[Y < y_0] = 0.70$. This problem can also be handled by converting it to one involving Z. We have

$$\begin{aligned} P[Y < y_0] &= P[Y - \mu < y_0 - \mu] && (6.30)\\ &= P\left[\frac{Y-\mu}{\sigma} < \frac{y_0-\mu}{\sigma}\right] && (6.31)\\ &= P\left[Z < \frac{y_0 - 50}{4}\right] && (6.32)\\ &= P[Z < z_0] && (6.33)\end{aligned}$$

where $z_0 = \frac{y_0-50}{4}$. We then search for a value of z_0 such that $P[Z < z_0] = 0.70$, and obtain $z_0 = 0.52$ from Table Z. We then solve for y_0 as follows:

$$\begin{aligned} z_0 &= \frac{y_0-50}{4} && (6.34)\\ 0.52 &= \frac{y_0-50}{4} && (6.35)\\ 4(0.52) &= y_0 - 50 && (6.36)\\ 2.08 &= y_0 - 50 && (6.37)\\ 2.08 + 50 &= y_0 && (6.38)\\ 52.08 &= y_0. && (6.39)\end{aligned}$$

So, $y_0 = 52.08$ is the answer. In general, one would have $z_0 = \frac{y_0-\mu}{\sigma}$, so $y_0 = \sigma z_0 + \mu$ for any σ and μ.

6.3 Expected values and variance for continuous distributions

We saw earlier how a theoretical mean, variance, and standard deviation could be calculated for a discrete random variable, using the concept of expectation and its probability distribution. The same concepts can be extended to continuous random variables and probability densities.

Let Y be a continuous random variable with some probability density. The expected value of Y, or its theoretical mean, is defined by the equation

$$E[Y] = \int_{-\infty}^{\infty} y f(y) dy \tag{6.40}$$

where $f(y)$ is the probability density of Y, and the integral is carried out over the interval $-\infty$ to ∞ (Mood et al. 1974). This equation is analogous to the definition of expected value for a discrete random variable, except that we use integration rather than summation to make the calculation.

Similar to discrete random variables, we can also define the theoretical variance of a continuous random variable using expectation. The variance of a continuous random variable Y is defined as

$$Var[Y] = \int_{-\infty}^{\infty} (y - E[Y])^2 f(y) dy. \tag{6.41}$$

We can directly calculate these quantities for the uniform distribution. Recall from calculus that $\int u du = u^2/2$. We therefore have

$$\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty} y f(y) dy = \int_a^b \frac{y}{b-a} dy && (6.42)\\
&= \frac{1}{b-a} \frac{y^2}{2} |_a^b = \frac{1}{b-a} \frac{b^2 - a^2}{2} && (6.43)\\
&= \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2} && (6.44)
\end{aligned}$$

Thus, the expected value (or theoretical mean) of a uniform random variable is located at the center of the interval, midway between a and b. It can also be shown using the above formula that the variance of the uniform distribution is

$$Var[Y] = \frac{(b-a)^2}{12} \tag{6.45}$$

The theoretical standard deviation is just the square root of this quantity.

What are these quantities for the normal distribution? Recall that the normal distribution is specified by the two parameters μ and σ^2. If $Y \sim N(\mu, \sigma^2)$, it can be shown (by evaluating the above integrals using the normal density) that

$$E[Y] = \mu \tag{6.46}$$

and

$$Var[Y] = \sigma^2. \tag{6.47}$$

Thus, the parameters μ and σ^2 for this distribution are the theoretical mean and variance $E[Y]$ and $Var[Y]$.

6.4 Continuous random variables and samples

Suppose we have a set of observations and want to determine if they can be modeled using the normal distribution. We now develop a graphical method of comparing these observed data with the pattern expected for the normal distribution, called a **normal quantile plot**. These plots exist for other continuous distributions as well, and are generally called quantile-quantile plots. The idea is to plot the quantiles for the observed data vs. the quantiles for the normal distribution, with the quantiles for the normal on the x-axis and the data quantiles on the y-axis. If the data are normally distributed, then this plot will resemble a straight diagonal line. This occurs because we are essentially plotting the quantiles for one normal distribution (the data) vs. the quantiles for the normal distribution itself (Wilk & Gnanadesikan 1968). This is like plotting the function $y = ax$, which is the equation of a line with slope a. See Chapter 3 for a review of quantiles such as the median, the 25% and 75% quartiles, and so forth.

Normal quantile plots are constructed as follows. Suppose we have five data points that take the values 2.1, 1.4, 3.9, 7.7, and 8.9. We first rank or order the data points from smallest to largest:

$$1.4, 2.1, 3.9, 7.7, 8.9. \tag{6.48}$$

We then determine a probability p corresponding to each data point using the formula $p = (r_i - 3/8)/(n + 1/4)$, where r_i is the rank order of the ith

data point and n is the sample size:

$$0.1190, 0.3095, 0.5, 0.6905, 0.8810. \tag{6.49}$$

The idea here is to associate a particular probability p with each data point, depending on its rank order. Note that the median of these data (the value 3.9) corresponds to $p = 0.5$. The values 3/8 and 1/4 in the formula are there to prevent p from taking the value 0 or 1 for the largest and smallest ranks. These are the values used by SAS for this purpose (SAS Institute Inc. 2014), although other ones have been suggested (Harter 1984, Makkonen 2008).

We then determine the quantiles of the standard normal distribution that correspond to the values of p for these data, using Table Z. For example, suppose we want to find a value z_0 such that $P[Z < z_0] = 0.5$, the median of the standard normal distribution. We see from Table Z that $z_0 = 0$ give the correct probability. For $p = 0.6905$, we find that $z_0 = 0.50$ gives close to the correct probability. We can similarly find the values of z_0 for the other values of p, to obtain:

$$-1.18, -0.50, 0, 0.50, 1.18. \tag{6.50}$$

The last step is to plot the rank ordered data vs. the normal quantiles, with the ordered data on the y-axis and corresponding normal quantiles on the x-axis. If the data are normally distributed, there will be a linear relationship between the observed data and the normal quantiles, and the normal quantile plot will be a straight line. If the data are non-normal, however, all manner of curved relationships are possible.

6.4.1 Elytra lengths - SAS demo

We previously examined a data set involving the elytra lengths of male and female *T. dubius* beetles and calculated various descriptive statistics using `proc univariate` (see Chapter 3). We now examine whether these data are normally-distributed using normal quantile plots. A normal quantile plot is requested by adding the command `qqplot` with the `normal` option to the program (see below). A histogram and fitted normal curve can also be generated using the `histogram` command with the `normal` option. Separate analyses are requested for male and female beetles using a `by` statement, because the two sexes differ in size and could also have potentially different distributions. We observe that the normal quantile plots for female beetles is close to linear, suggesting a normal distribution, while the males show some curvature.

SAS Program

```
* normal_quantile_plot.sas;
options pageno=1 linesize=80;
title 'Fitting the normal to elytra data';
data elytra;
    input sex $ length;
    datalines;
M   4.9
F   5.2
M   4.9
F   4.2
F   5.7

etc.

M   5.1
F   4.4
M   4.8
M   4.6
F   3.7
;
run;
* Descriptive statistics, histograms, and normal quantile plots;
proc univariate plots data=elytra;
    * Separate analyses for each sex;
    class sex;
    var length;
    histogram length/ vscale=count normal(w=3) wbarline=3 waxis=3 height=4;
    qqplot length / normal waxis=3 height=4;
    symbol1 h=3;
run;
quit;
```

SAS Output

```
                     Fitting the normal to elytra data                       1
                                                  11:03 Thursday, May 13, 2010

                          The UNIVARIATE Procedure
                             Variable:  length
                                  sex = F

                                  Moments

        N                          60    Sum Weights                 60
        Mean                     4.94    Sum Observations         296.4
        Std Deviation      0.48544929    Variance            0.23566102
        Skewness            -0.521146    Kurtosis            0.16125847
        Uncorrected SS        1478.12    Corrected SS            13.904
        Coeff Variation    9.82690878    Std Error Mean      0.06267123


                        Basic Statistical Measures

              Location                    Variability

             Mean     4.940000     Std Deviation            0.48545
             Median   5.000000     Variance                 0.23566
             Mode     5.200000     Range                    2.20000
                                   Interquartile Range      0.70000


                        Tests for Location: Mu0=0

               Test           -Statistic-    -----p Value------

               Student's t    t  78.82404    Pr > |t|    <.0001
               Sign           M        30    Pr >= |M|   <.0001
               Signed Rank    S       915    Pr >= |S|   <.0001


                         Quantiles (Definition 5)

                          Quantile      Estimate

                          100% Max           5.9
                          99%                5.9
                          95%                5.7
```

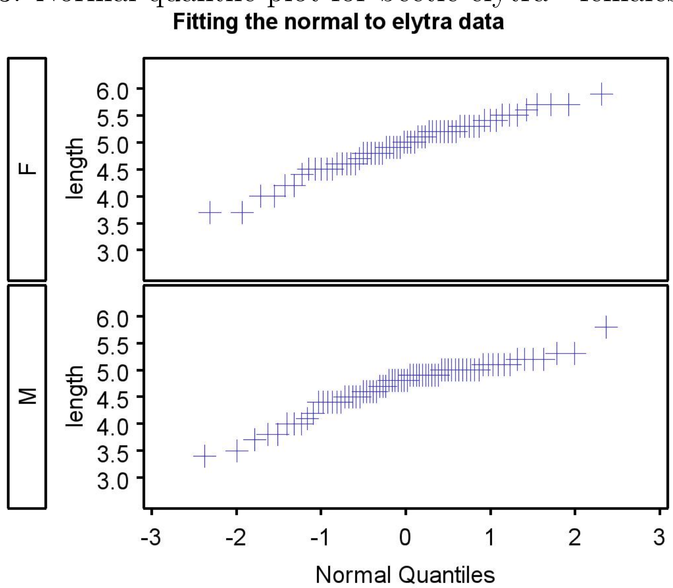
```
               90%                5.5
               75% Q3             5.3
               50% Median         5.0
               25% Q1             4.6
               10%                4.3
               5%                 4.0
               1%                 3.7
               0% Min             3.7


         Fitting the normal to elytra data                   4
                                         11:03 Thursday, May 13, 2010

                 The UNIVARIATE Procedure
                    Variable:  length
                         sex = M

                         Moments

N                         70    Sum Weights                 70
Mean              4.71285714    Sum Observations         329.9
Std Deviation     0.44977335    Variance            0.20229607
Skewness           -0.896502    Kurtosis            1.00307174
Uncorrected SS       1568.73    Corrected SS        13.9584286
Coeff Variation    9.5435388    Std Error Mean       0.0537582


                Basic Statistical Measures

          Location                    Variability

     Mean     4.712857     Std Deviation            0.44977
     Median   4.800000     Variance                 0.20230
     Mode     5.000000     Range                    2.40000
                           Interquartile Range      0.50000


                Tests for Location: Mu0=0

        Test           -Statistic-    -----p Value------

        Student's t    t  87.66769    Pr > |t|    <.0001
        Sign           M        35    Pr >= |M|   <.0001
        Signed Rank    S    1242.5    Pr >= |S|   <.0001
```

```
                   Quantiles (Definition 5)

                    Quantile      Estimate

                    100% Max          5.80
                    99%               5.80
                    95%               5.20
                    90%               5.15
                    75% Q3            5.00
                    50% Median        4.80
                    25% Q1            4.50
                    10%               4.00
                    5%                3.80
                    1%                3.40
                    0% Min            3.40


          Fitting the normal to elytra data                        7
                                 11:03 Thursday, May 13, 2010

               The UNIVARIATE Procedure
                       sex = F
          Fitted Normal Distribution for length

            Parameters for Normal Distribution

              Parameter   Symbol   Estimate

              Mean        Mu           4.94
              Std Dev     Sigma    0.485449


        Goodness-of-Fit Tests for Normal Distribution

Test                  ----Statistic-----   ------p Value------

Kolmogorov-Smirnov    D       0.10387776   Pr > D        0.105
Cramer-von Mises      W-Sq    0.07705508   Pr > W-Sq     0.228
Anderson-Darling      A-Sq    0.50377430   Pr > A-Sq     0.206


             Quantiles for Normal Distribution

                        ------Quantile------
```

```
Percent   Observed   Estimated

    1.0    3.70000     3.81068
    5.0    4.00000     4.14151
   10.0    4.30000     4.31787
   25.0    4.60000     4.61257
   50.0    5.00000     4.94000
   75.0    5.30000     5.26743
   90.0    5.50000     5.56213
   95.0    5.70000     5.73849
   99.0    5.90000     6.06932


             Fitting the normal to elytra data                             8
                                        11:03 Thursday, May 13, 2010

                The UNIVARIATE Procedure
                        sex = M
           Fitted Normal Distribution for length

             Parameters for Normal Distribution

               Parameter   Symbol   Estimate

               Mean        Mu       4.712857
               Std Dev     Sigma    0.449773


          Goodness-of-Fit Tests for Normal Distribution

Test                  ----Statistic-----   ------p Value------

Kolmogorov-Smirnov    D       0.16252783   Pr > D       <0.010
Cramer-von Mises      W-Sq    0.34087445   Pr > W-Sq    <0.005
Anderson-Darling      A-Sq    1.99478432   Pr > A-Sq    <0.005


              Quantiles for Normal Distribution

                    ------Quantile------
               Percent   Observed   Estimated

                   1.0    3.40000     3.66653
                   5.0    3.80000     3.97305
                  10.0    4.00000     4.13645
```

25.0	4.50000	4.40949
50.0	4.80000	4.71286
75.0	5.00000	5.01622
90.0	5.15000	5.28926
95.0	5.20000	5.45267
99.0	5.80000	5.75919

Figure 6.13: Normal quantile plot for beetle elytra - females and males



6.4.2 Development time - SAS demo

We now examine a data set involving the development time of *T. dubius* beetles in various stages, in particular the time from the larval to prepupal stage, and then from the prepupal to adult stage (Reeve et al. 2003). See program below for details of this analysis. We see that the normal quantile plots for both stages are quite nonlinear, suggesting a distribution different from normal. This is a reflection of the skewed distributions of development time we saw earlier for these data (Chapter 3). Skewed and nonnormal distributions are a common feature of insect development data (Wagner et al. 1984).

SAS Program

```
* normal_quantile_plot_2.sas;
options pageno=1 linesize=80;
title 'Fitting the normal to development data';
data devel_time;
    input time_pp time_adult;
    datalines;
34  65
31  48
29   .
30  55
32  62

etc.

29   .
29 108
31 103
33   .
29  92
;
run;
* Descriptive statistics, histograms, and normal quantile plots;
proc univariate plots data=devel_time;
    var time_pp time_adult;
    histogram time_pp time_adult / vscale=count normal(w=3) wbarline=3 waxis=3 height=4;
    qqplot time_pp time_adult / normal waxis=3 height=4;
    symbol1 h=3;
run;
quit;
```

SAS Output

```
                 Fitting the normal to development data                 1
                                              08:08 Thursday, April 29, 2010

                        The UNIVARIATE Procedure
                           Variable:  time_pp

                                Moments

    N                          96    Sum Weights                 96
    Mean               31.3541667    Sum Observations          3010
    Std Deviation      3.32764866    Variance            11.0732456
    Skewness           0.75038358    Kurtosis            0.04666776
    Uncorrected SS          95428    Corrected SS        1051.95833
    Coeff Variation    10.6130987    Std Error Mean      0.33962672


                       Basic Statistical Measures

             Location                    Variability

         Mean     31.35417     Std Deviation            3.32765
         Median   31.00000     Variance                11.07325
         Mode     30.00000     Range                   14.00000
                               Interquartile Range      5.00000


                       Tests for Location: Mu0=0

           Test           -Statistic-    -----p Value------

           Student's t    t  92.31949    Pr > |t|    <.0001
           Sign           M        48    Pr >= |M|   <.0001
           Signed Rank    S      2328    Pr >= |S|   <.0001


                        Quantiles (Definition 5)

                         Quantile      Estimate

                         100% Max            41
                         99%                 41
                         95%                 39
                         90%                 36
```

```
         75% Q3              34
         50% Median          31
         25% Q1              29
         10%                 27
         5%                  27
         1%                  27
         0% Min              27


     Fitting the normal to development data                    4
                                08:08 Thursday, April 29, 2010

            The UNIVARIATE Procedure
     Fitted Normal Distribution for time_pp

       Parameters for Normal Distribution

         Parameter   Symbol   Estimate

         Mean        Mu       31.35417
         Std Dev     Sigma    3.327649


   Goodness-of-Fit Tests for Normal Distribution

Test                  ----Statistic-----   ------p Value------

Kolmogorov-Smirnov    D       0.13138957   Pr > D       <0.010
Cramer-von Mises      W-Sq    0.26720735   Pr > W-Sq    <0.005
Anderson-Darling      A-Sq    1.73548398   Pr > A-Sq    <0.005


        Quantiles for Normal Distribution

                  ------Quantile------
         Percent   Observed   Estimated

             1.0    27.0000     23.6129
             5.0    27.0000     25.8807
            10.0    27.0000     27.0896
            25.0    29.0000     29.1097
            50.0    31.0000     31.3542
            75.0    34.0000     33.5986
            90.0    36.0000     35.6187
```

```
                   95.0    39.0000     36.8277
                   99.0    41.0000     39.0954


              Fitting the normal to development data              5
                                    08:08 Thursday, April 29, 2010

                   The UNIVARIATE Procedure
                    Variable:  time_adult

                           Moments

N                          68    Sum Weights                 68
Mean               75.3529412    Sum Observations          5124
Std Deviation      26.3465791    Variance             694.14223
Skewness           0.51461555    Kurtosis            -0.6244048
Uncorrected SS         432616    Corrected SS        46507.5294
Coeff Variation    34.9642346    Std Error Mean      3.19499201


                  Basic Statistical Measures

         Location                    Variability

     Mean     75.35294     Std Deviation            26.34658
     Median   68.00000     Variance                694.14223
     Mode     42.00000     Range                   105.00000
                           Interquartile Range      46.50000


                  Tests for Location: Mu0=0

        Test           -Statistic-    -----p Value------

        Student's t    t   23.5847    Pr > |t|    <.0001
        Sign           M        34    Pr >= |M|   <.0001
        Signed Rank    S      1173    Pr >= |S|   <.0001


                    Quantiles (Definition 5)

                    Quantile      Estimate

                    100% Max         147.0
                    99%              147.0
```

```
95%              116.0
90%              110.0
75% Q3            99.0
50% Median        68.0
25% Q1            52.5
10%               43.0
5%                42.0
1%                42.0
0% Min            42.0


           Fitting the normal to development data                    8
                                       08:08 Thursday, April 29, 2010

                 The UNIVARIATE Procedure
          Fitted Normal Distribution for time_adult

              Parameters for Normal Distribution

                Parameter   Symbol   Estimate

                Mean        Mu       75.35294
                Std Dev     Sigma    26.34658


          Goodness-of-Fit Tests for Normal Distribution

Test                  ----Statistic-----   ------p Value------

Kolmogorov-Smirnov    D       0.12461617   Pr > D       <0.010
Cramer-von Mises      W-Sq    0.22866485   Pr > W-Sq    <0.005
Anderson-Darling      A-Sq    1.43281773   Pr > A-Sq    <0.005


              Quantiles for Normal Distribution

                         -------Quantile------
               Percent    Observed   Estimated

                   1.0     42.0000     14.0616
                   5.0     42.0000     32.0167
                  10.0     43.0000     41.5884
                  25.0     52.5000     57.5824
                  50.0     68.0000     75.3529
```

```
75.0     99.0000     93.1234
90.0    110.0000    109.1174
95.0    116.0000    118.6892
99.0    147.0000    136.6442
```

Figure 6.14: Development time- larval to prepupal stage

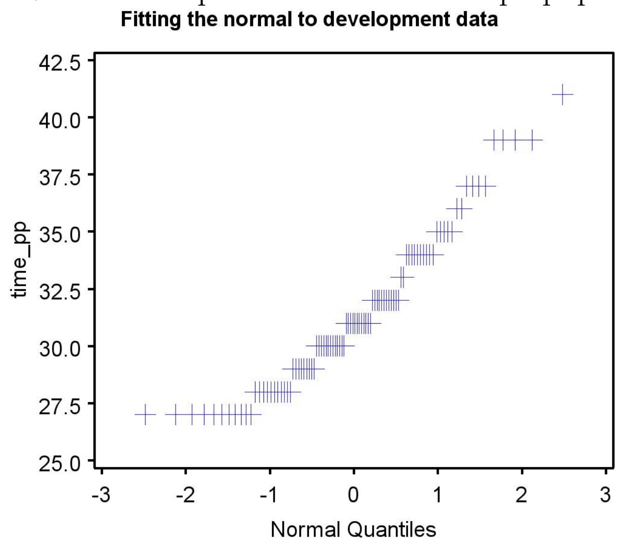
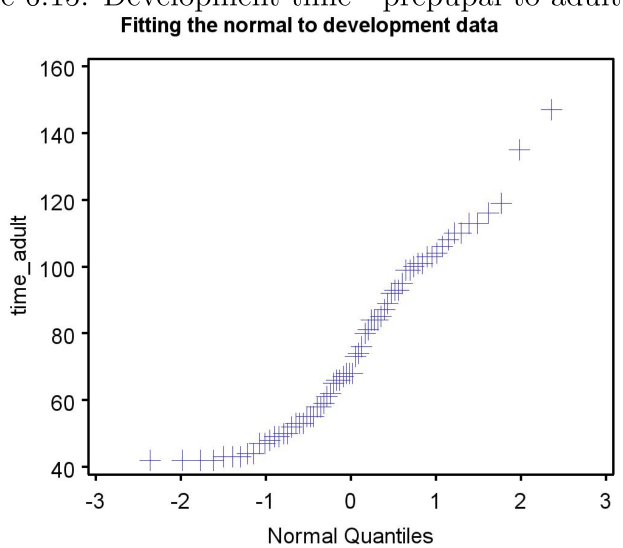


Figure 6.15: Development time - prepupal to adult stage



6.5 References

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6.6 Problems

1. A random variable Y has a uniform probability density with $a = 0$ and $b = 2$.
   (a) What is the expected value of Y, or $E[Y]$? What is the variance of Y, or $Var[Y]$?
   (b) What are the 25%, 50%, and 80% quantiles or percentiles of Y?
   (c) Find the probability that $Y < 0.05$.
   (d) Find a symmetric interval centered around $y = 1$ that has a probability of 0.95.
2. Suppose that Y has a normal distribution with $\mu = 1$ and $\sigma^2 = 3$, or $Y \sim N(1, 3)$. Find the following quantities using Table Z.
   (a) The probability that $Y > 2$.
   (b) The probability that $1 < Y < 3$.
   (c) The probability that $Y < 0.5$.
   (d) The probability that Y is not inside the interval given in b.
   (e) A value of y_0 such that the probability that $Y < y_0$ is 0.9.
3. Suppose that Y has a normal distribution with $\mu = 2$ and $\sigma^2 = 4$, or $Y \sim N(2, 4)$. Find the following quantities using Table Z:
   (a) The probability that $Y < 2.5$.
   (b) The probability that $0.5 < Y < 2.5$.
   (c) The probability that $Y < 1$.
   (d) The probability that Y is not inside the interval given in b.
   (e) A value of y_0 such that the probability that $Y < y_0$ is 0.4.