Chapter 9

Confidence Intervals

In the preceding chapter, we examined the maximum likelihood method for estimating the parameters of a statistical population, using a random sample from that population. For example, if we have a sample from a population with a normal distribution, we can estimate the parameter μ of this population using the sample mean $\bar{Y}$. We will now examine a common method for characterizing the precision of these estimates, known as **confidence intervals**. Given an estimate $\bar{Y}$ of μ, say, we will learn how to calculate an interval that will contain the true population μ with a certain probability. A narrow interval indicates the parameter μ is reliably estimated, while a broad one indicates substantial uncertainty as to its value.

9.1 Preliminaries to confidence intervals

We now discuss some material that is essential for the construction of confidence intervals and later in hypothesis testing. We first review some results from Chapter 8 on parameter estimation for the normal distribution, then derive some new results. We then examine some distributions associated with sampling from the normal distributions, not surprisingly called **sampling distributions**.

9.1.1 Parameters and estimates

Confidence intervals are based on estimates of population parameters, such as μ and σ^2 for populations with a normal distribution. Our previous results on parameter estimation suggest that $\bar{Y}$ and s^2 are reasonable estimators of μ and σ^2. The sample standard deviation $s = \sqrt{s^2}$ is typically used to estimate the population standard deviation σ.

We also want to estimate the variance and standard deviation of the sample mean $\bar{Y}$. Recall that for a random sample Y_1, Y_2, ... Y_n with any distribution,

$$Var[\bar{Y}] = \frac{Var[Y_i]}{n} \tag{9.1}$$

where $Var[Y_i]$ is the variance of Y_i (Chapter 7). For a random sample where the observations are normal, this translates to

$$Var[\bar{Y}] = \frac{\sigma^2}{n} \tag{9.2}$$

because $Var[Y_i] = \sigma^2$ for the normal. If we use s^2 to estimate σ^2, we can therefore estimate $Var[\bar{Y}]$ using s^2/n and $\sigma/\sqrt{n}$ using $s/\sqrt{n}$.

The table below summarizes the different parameters, their estimators, and common terminology for these quantities:

Table 9.1: Parameters and their estimators

Parameter	Estimator	Terminology
μ	$\bar{Y}$	Sample mean
σ^2	s^2	Sample variance
σ	s	Sample standard deviation
$\frac{\sigma^2}{n}$	$\frac{s^2}{n}$	Sample variance of the mean
$\frac{\sigma}{\sqrt{n}}$	$\frac{s}{\sqrt{n}}$	Standard error of the mean

Recall that the term standard error always refers to the standard deviation of a statistic, such as $\bar{Y}$. The term standard deviation used without qualification usually means the standard deviation s of items in a random sample from a population.

9.1.2 Sampling distributions

In this section, we will first examine the probability distribution of the estimator $\bar{Y}$. We then examine the distributions of some quantities involving $\bar{Y}$ and the sample variance s^2, known as sampling distributions. These sampling distributions will be used to construct confidence intervals and also play an important role in hypothesis testing (Chapter 10).

Distribution of $\bar{Y}$

Suppose we have a random sample $Y_1, Y_2, ..., Y_n$ from a statistical population with a normal distribution, in particular that $Y_i \sim N(\mu, \sigma^2)$ and are independent of each other. It can be shown that

$$\bar{Y} \sim N\left(\mu, \frac{\sigma^2}{n}\right). \tag{9.3}$$

Thus, the sample mean of normal observations also has a normal distribution with the same mean μ, but with variance equal to σ^2/n, not σ^2 (Mood *et al.* 1974).

Note that the distribution of $\bar{Y}$ will be approximately normal for any distribution provided n is large, thanks to the central limit theorem (see Chapter 7). Thus, for large sample sizes we have $\bar{Y} \sim N(E[Y], Var[Y]/n)$ for any probability distribution. This result has important statistical implications. **Confidence intervals and hypothesis testing procedures often assume that $\bar{Y}$ is normally distributed, and this will be approximately true if n is sufficiently large.** These statistical procedures are therefore robust to departures from normality in the data for large n.

We also learned earlier that if $Y \sim N(\mu, \sigma^2)$, then the transformed variable $(Y - \mu)/\sigma$ has a standard normal distribution, or $(Y - \mu)/\sigma = Z \sim N(0, 1)$. Combining these two results, we find that

$$\frac{\bar{Y} - \mu}{\sqrt{\sigma^2/n}} = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1) \tag{9.4}$$

Thus, the quantity $\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}$ has a standard normal distribution. We will use this sampling distribution to obtain a confidence interval for μ, for the case where σ^2 is known from other information.

We will also need to find certain intervals with a specified probability using the standard normal distribution, in order to construct confidence intervals. In general, we will need to find a positive value c such that

$$P[-c_\alpha < Z < c_\alpha] = 1 - \alpha \tag{9.5}$$

for this purpose, where typically $\alpha = 0.05$ or 0.01. The values of c_α that satisfy this probability are often called **critical values**, a term that also applies to other probability distributions. We use the notation c_α because this quantity depends on the value of α. To find c_α, we first express this probability in terms of Table Z. We have

$$P[-c_\alpha < Z < c_\alpha] = P[Z < c_\alpha] - P[Z < -c_\alpha] \tag{9.6}$$
$$= P[Z < c_\alpha] - (1 - P[Z < c_\alpha]) \tag{9.7}$$
$$= 2P[Z < c_\alpha] - 1. \tag{9.8}$$

If we set $2P[Z < c_\alpha] - 1 = 1 - \alpha$ and rearrange, we get

$$P[Z < c_\alpha] = (2 - \alpha)/2 = 1 - \alpha/2. \tag{9.9}$$

Therefore, we examine Table Z for a value of c_α such that $P[Z < c_\alpha] = 1 - \alpha/2$. For $\alpha = 0.05$, we would look for $c_{0.05}$ such that $P[Z < c_{0.05}] = 1 - 0.05/2 = 0.975$ and find that $c_{0.05} = 1.96$ is the answer. Similarly, for $\alpha = 0.01$ we seek $c_{0.01}$ such that $P[Z < c_{0.01}] = 1 - 0.01/2 = 0.995$. There is no value in Table Z that gives quite this probability, although we can see 2.57 and 2.58 are close. The exact answer is $c_{0.01} = 2.576$.

t distribution

Another important sampling distribution is the t distribution. This distribution has a single parameter, called the degrees of freedom, that governs the shape of the distribution. It can be shown that the quantity

$$\frac{\bar{Y} - \mu}{s/\sqrt{n}} \sim t_{n-1} \tag{9.10}$$

(Mood *et al.* 1974). Here the symbol 't_{n-1}' stands for the t distribution with $n-1$ degrees of freedom, where n is the sample size in $\bar{Y}$. Degrees of freedom is often abbreviated as '*df*'.

The t distribution resembles the standard normal distribution in being bell-shaped, except that it has more probability in the tails and less in the center of the distribution (Fig. 9.1). Roughly speaking, the t distribution has heavier tails than the normal because $\bar{Y}$ and s are both random quantities in Eq. 9.10, making their ratio more variable than for Eq. 9.4 where only $\bar{Y}$ is random. However, as $n \to \infty$ the t distribution does approach the standard normal distribution. We will use this sampling distribution to obtain a confidence interval for μ, when σ^2 is estimated using the sample variance s^2.

What is the origin of the term degrees of freedom? Recall that the sample standard deviation s is obtained from the sample variance, calculated using the formula

$$s^2 = \frac{\Sigma_{i=1}^n (Y_i - \bar{Y})^2}{n-1}. \tag{9.11}$$

Notice that the sample variance s^2 is composed of terms of the form $Y_i - \bar{Y}$. Although there are n of these terms, they also sum to zero ($\Sigma_i^n (Y_i - \bar{Y}) = 0$). This implies that if $n-1$ terms are known, we can always determine the remaining term because of this relationship, implying there are really only $n-1$ free, independent terms in s^2 (Mood et al. 1974). Hence the name degrees of freedom.

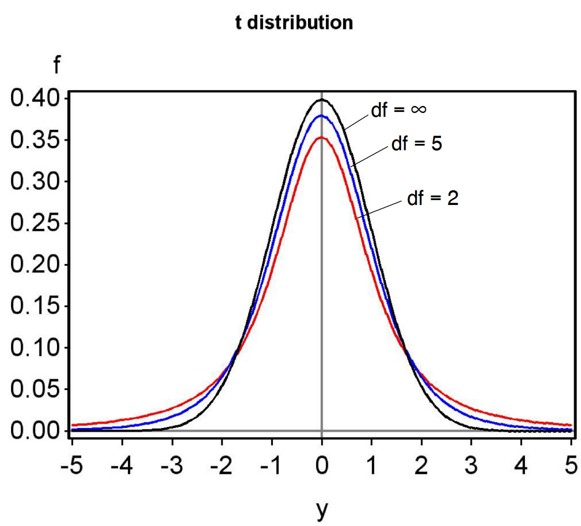

Figure 9.1: Plot of the t distribution for different degrees of freedom

Table T gives the quantiles of the t distribution for different values of the degrees of freedom and the cumulative probability p. We will also need to find intervals of the form

$$P[-c_{\alpha,df} < T < c_{\alpha,df}] = 1 - \alpha, \tag{9.12}$$

where $c_{\alpha,df}$ is a positive number, T has a t distribution, for $\alpha = 0.05$ or 0.01. We use the notation $c_{\alpha,df}$ because this quantity will depend on both α and the degrees of freedom. We proceed as before by expressing this probability in terms of Table T. We have

$$P[-c_{\alpha,df} < T < c_{\alpha,df}] = P[T < c_{\alpha,df}] - P[T < -c_{\alpha,df}] \tag{9.13}$$
$$= P[T < c_{\alpha,df}] - (1 - P[T < c_{\alpha,df}]) \tag{9.14}$$
$$= 2P[T < c_{\alpha,df}] - 1. \tag{9.15}$$

If we set $2P[T < c_{\alpha,df}] - 1 = 1 - \alpha$ and rearrange, we get

$$2(1 - P[T < c_{\alpha,df}]) = \alpha. \tag{9.16}$$

Because $P[T < c_{\alpha,df}]$ is essentially p for this table, we simply look across the row corresponding to $2(1-p)$ at the top and find the column corresponding to α. For $\alpha = 0.05$, we see that for $df = 10$ the answer is $c_{0.05,10} = 2.228$. For $\alpha = 0.01$ and $df = 10$, the answer is $c_{0.01,10} = 3.169$.

χ^2 distribution

One other common sampling distribution is the χ^2 (chi-square) distribution, which also has a parameter called the degrees of freedom. It can be shown that the quantity

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1} \tag{9.17}$$

(Mood *et al.* 1974). Here the symbol 'χ^2_{n-1}' stands for a χ^2 distribution with $n-1$ degrees of freedom. The degrees of freedom parameter controls the shape of the χ^2 distribution (Fig. 9.2). The χ^2 distribution is only defined for positive values, because s^2 is always positive, and its distribution shifts to the right (large values become more likely) as n and the degrees of freedom increases. We will use this sampling distribution to obtain a confidence interval for σ^2 and σ.

Table C gives the quantiles of the χ^2 distribution for different values of the degrees of freedom and the cumulative probability p. We will need to find the probabilities for certain intervals, but this is more complicated with the χ^2 distribution because it is asymmetrical, unlike the normal or t distributions. In this case, we want to find two positive numbers $c_{\alpha/2,df}$ and $c_{1-\alpha/2,df}$ such that

$$P[c_{\alpha/2,df} < X < c_{1-\alpha/2,df}] = 1 - \alpha, \tag{9.18}$$

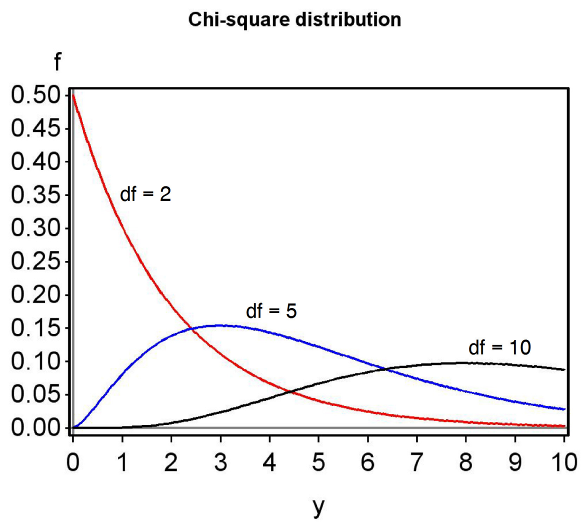

Figure 9.2: Plot of the χ^2 distribution for different degrees of freedom

where X has a χ^2 distribution and $\alpha = 0.05$ or $\alpha = 0.01$. The subscripts $\alpha/2$ and $1 - \alpha/2$ for c essentially correspond to values of p in Table C. This gives the correct probability because

$$P[c_{\alpha/2,df} < X < c_{1-\alpha/2,df}] = P[X < c_{1-\alpha/2,df}] - P[X < c_{\alpha/2,df}] \tag{9.19}$$
$$= 1 - \alpha/2 - \alpha/2 = 1 - \alpha. \tag{9.20}$$

To see how these values are obtained from Table C, suppose that $\alpha = 0.05$ and $df = 10$. To find $c_{\alpha/2,df} = c_{0.05/2,10} = c_{0.025,10}$, we look in the column for $p = 0.025$ and row for $df = 10$, and obtain $c_{0.025,10} = 3.247$. To find $c_{1-\alpha/2,df} = c_{1-0.05/2,10} = c_{0.975,10}$, we look in the column for $p = 0.975$ and row for $df = 10$, and obtain $c_{0.975,10} = 20.483$.

Now suppose that $\alpha = 0.01$. Using the same technique, we find that $c_{\alpha/2,df} = c_{0.01/2,10} = c_{0.005,10} = 2.156$, and $c_{1-\alpha/2,df} = c_{1-0.01/2,10} = c_{0.995,10} = 25.188$.

9.2 Confidence intervals

We now have the information needed to calculate confidence intervals. We will begin with a simple but unrealistic case, finding a confidence interval for

μ when σ^2 is known through other means. This case is unrealistic because σ^2 is almost always estimated from the data, but the calculations are simple and illustrate a general method for finding confidence intervals. We then turn to finding a confidence intervals for μ, and then σ^2, where all parameters are estimated from the data.

9.2.1 Confidence intervals for μ when σ^2 is known

We will use the fact that the quantity $\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}$ has a standard normal distribution to find a confidence interval for μ. Suppose that α is given and we have found c_α such that

$$P\left[-c_\alpha < Z < c_\alpha\right] = 1-\alpha. \tag{9.21}$$

(see previous section). Substituting $\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}$ for Z we obtain

$$P\left[-c_\alpha < \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} < c_\alpha\right] = 1-\alpha. \tag{9.22}$$

Multiplying both sides by $\sigma/\sqrt{n}$ gives you

$$P\left[-c_\alpha \frac{\sigma}{\sqrt{n}} < \bar{Y}-\mu < c_\alpha \frac{\sigma}{\sqrt{n}}\right] = 1-\alpha. \tag{9.23}$$

Multiplying all parts inside the brackets by -1 reverses the signs and inequalities to give

$$P\left[c_\alpha \frac{\sigma}{\sqrt{n}} > \mu-\bar{Y} > -c_\alpha \frac{\sigma}{\sqrt{n}}\right] = 1-\alpha. \tag{9.24}$$

We now add to $\bar{Y}$ to all parts inside the brackets to give

$$P\left[\bar{Y}+c_\alpha \frac{\sigma}{\sqrt{n}} > \mu > \bar{Y}-c_\alpha \frac{\sigma}{\sqrt{n}}\right] = 1-\alpha, \tag{9.25}$$

or equivalently

$$P\left[\bar{Y}-c_\alpha \frac{\sigma}{\sqrt{n}} < \mu < \bar{Y}+c_\alpha \frac{\sigma}{\sqrt{n}}\right] = 1-\alpha. \tag{9.26}$$

We call the terms $\bar{Y}-c_\alpha \frac{\sigma}{\sqrt{n}}$ and $\bar{Y}+c_\alpha \frac{\sigma}{\sqrt{n}}$ the lower and upper $100(1-\alpha)\%$ confidence limits for μ (Mood et al. 1974). Confidence intervals are often reported in the form $(\bar{Y} - c_\alpha \frac{\sigma}{\sqrt{n}}, \bar{Y} + c_\alpha \frac{\sigma}{\sqrt{n}})$. Note that the center of the confidence interval is at $\bar{Y}$, our estimate of μ. This interval would be expected to include the true value of μ with a probability of $1 - \alpha$, because this was the probability set in Eq. 9.21.

It is common practice to set $\alpha = 0.05$, which corresponds to a $100(1 - 0.05)\% = 95\%$ confidence interval. For this case, we would have $c_\alpha = c_{0.05} = 1.96$ (see previous section). Therefore, the 95% confidence interval would be

$$(\bar{Y} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{Y} + 1.96\frac{\sigma}{\sqrt{n}}). \tag{9.27}$$

We would expect this interval to include the true μ with a probability of 0.95, or 95% of the time. However, it follows that the interval will miss μ with a probability of 0.05, or 5% of the time. **This is an important feature of confidence intervals - they will often but not always enclose the true parameter value for the population, with the probability set by α.**

If we wanted to be more certain of including μ, we could choose a smaller α, say $\alpha = 0.01$, which corresponds to a $100(1 - 0.01)\% = 99\%$ confidence interval. Here we have $c_\alpha = c_{0.01} = 2.576$, and so the 99% confidence interval would be

$$(\bar{Y} - 2.576\frac{\sigma}{\sqrt{n}}, \bar{Y} + 2.576\frac{\sigma}{\sqrt{n}}). \tag{9.28}$$

A 99% confidence interval will necessarily be broader than a 95% one, because it is constructed to have a higher probability of including μ.

Confidence intervals - sample calculation

Suppose we have a sample of $n = 10$ elytra from female *T. dubius* beetles (see Chapter 3 for a description of these data), yielding the values listed below:

```
5.0 5.1 5.2 5.9 4.8 5.5 4.8 5.1 5.0 5.1
```

For this sample, we calculate that $\bar{Y} = 5.150$. Suppose we have *a priori* knowledge that $\sigma = 0.3$, although that would be rare in practice. We will calculate a 95% and 99% confidence interval for μ.

The formula for a 95% confidence interval is

$$(\bar{Y} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{Y} + 1.96\frac{\sigma}{\sqrt{n}}). \tag{9.29}$$

Substituting $n = 10, \bar{Y} = 5.150$, and $\sigma = 0.3$ in the above formula, we obtain

$$(5.150 - 1.96\frac{0.3}{\sqrt{10}}, 5.150 + 1.96\frac{0.3}{\sqrt{10}}), \tag{9.30}$$

or

$$(5.150 - 0.186, 5.150 + 0.186), \tag{9.31}$$

or

$$(4.964, 5.336). \tag{9.32}$$

So, the 95% confidence interval for μ is $(4.964, 5.336)$.

For a 99% confidence interval, we use the formula

$$(\bar{Y} - 2.576\frac{\sigma}{\sqrt{n}}, \bar{Y} + 2.576\frac{\sigma}{\sqrt{n}}). \tag{9.33}$$

Substituting as before, we obtain

$$(5.150 - 2.576\frac{0.3}{\sqrt{10}}, 5.150 + 2.576\frac{0.3}{\sqrt{10}}), \tag{9.34}$$

or

$$(5.150 - 0.244, 5.150 + 0.244), \tag{9.35}$$

or

$$(4.906, 5.394). \tag{9.36}$$

The 99% confidence interval is therefore $(4.906, 5.394)$. Note that the 99% confidence interval is broader than the 95% one, because its lower limit is lower and upper limit higher.

9.2.2 Confidence intervals for μ when σ^2 is estimated

Confidence intervals for μ can also be derived when σ^2 is estimated using the sample variance s^2, as will usually be the case in practice. We will make use of the fact that

$$\frac{\bar{Y} - \mu}{s/\sqrt{n}} \sim t_{n-1}. \tag{9.37}$$

Using Table T, we can find a value of $c_{\alpha,n-1}$ for $n-1$ degrees of freedom such that the following equation is true:

$$P\left[-c_{\alpha,n-1} < \frac{\bar{Y} - \mu}{s/\sqrt{n}} < c_{\alpha,n-1}\right] = 1 - \alpha. \tag{9.38}$$

Rearranging this equation using the same procedures as before, we obtain

$$P\left[\bar{Y} - c_{\alpha,n-1}\frac{s}{\sqrt{n}} < \mu < \bar{Y} + c_{\alpha,n-1}\frac{s}{\sqrt{n}}\right] = 1 - \alpha. \tag{9.39}$$

The terms $\bar{Y} - c_{\alpha,n-1}\frac{s}{\sqrt{n}}$ and $\bar{Y} + c_{\alpha,n-1}\frac{s}{\sqrt{n}}$ are the lower and upper $100(1-\alpha)\%$ confidence limits for μ (Mood et al. 1974). The interval would be reported in the form $(\bar{Y} - c_{\alpha,n-1}\frac{s}{\sqrt{n}}, \bar{Y} + c_{\alpha,n-1}\frac{s}{\sqrt{n}})$. The center of the confidence interval is located at $\bar{Y}$, the estimate of μ.

For example, if we let $\alpha = 0.05$ this corresponds to a 95% confidence interval of the form

$$(\bar{Y} - c_{0.05,n-1}\frac{s}{\sqrt{n}}, \bar{Y} + c_{0.05,n-1}\frac{s}{\sqrt{n}}). \tag{9.40}$$

The value of $c_{0.05,n-1}$ would need to be determined from Table T, using the column for $2(1-p) = \alpha = 0.05$ and the row for $n-1$ degrees freedom.

For $\alpha = 0.01$, we obtain a 99% confidence interval of the form

$$(\bar{Y} - c_{0.01,n-1}\frac{s}{\sqrt{n}}, \bar{Y} + c_{0.01,n-1}\frac{s}{\sqrt{n}}). \tag{9.41}$$

In this case, we would use the column for $2(1-p) = \alpha = 0.01$ to find the value of $c_{0.01,n-1}$, using $n-1$ degrees freedom.

Confidence interval for μ - sample calculation

We return to the elytra data set, for which we previously calculated that $\bar{Y} = 5.150$, $s^2 = 0.109$, and $s = 0.331$ for $n = 10$. We will calculate 95% and 99% confidence intervals for μ.

The formula for a 95% confidence interval is

$$(\bar{Y} - c_{0.05,n-1}\frac{s}{\sqrt{n}}, \bar{Y} + c_{0.05,n-1}\frac{s}{\sqrt{n}}). \tag{9.42}$$

For $n = 10$, we have $df = n - 1 = 10 - 1 = 9$. For a 95% confidence interval, we therefore look up $c_{0.05,n-1} = c_{0.05,9}$ using the column for $2(1-p) = 0.05$ in Table T, choosing the value for 9 degrees of freedom. We obtain $c_{0.05,9} = 2.262$. Substituting $n = 10, \bar{Y} = 5.150, s = 0.331$, and $c_{0.05,9} = 2.262$ in the above formula, we obtain

$$(5.150 - 2.262\frac{0.331}{\sqrt{10}}, 5.150 + 2.262\frac{0.331}{\sqrt{10}}), \tag{9.43}$$

or

$$(5.150 - 0.237, 5.150 + 0.237), \tag{9.44}$$

or

$$(4.913, 5.387). \tag{9.45}$$

So, the 95% confidence interval for μ is $(4.913, 5.387)$. For a 99% confidence interval, we find $c_{0.01,n-1} = c_{0.01,9}$ for $2(1-p) = 0.01$ and 9 degrees of freedom in Table T, obtaining $c_{0.01,9} = 3.250$. Substituting this value in the above formula, we obtain

$$(5.150 - 3.250\frac{0.331}{\sqrt{10}}, 5.150 + 3.250\frac{0.331}{\sqrt{10}}), \tag{9.46}$$

or

$$(5.150 - 0.340, 5.150 + 0.340), \tag{9.47}$$

or

$$(4.810, 5.490). \tag{9.48}$$

The 99% confidence interval is therefore $(4.810, 5.490)$, and as expected is broader than the 95% one.

9.2.3 Confidence intervals for σ^2 and σ

Confidence intervals for σ^2 and σ can also be derived, using the fact that

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1} \tag{9.49}$$

Using Table C for the χ^2 distribution, we can find values $c_{\alpha/2,n-1}$ and $c_{1-\alpha/2,n-1}$ for $n-1$ degrees of freedom such that the following equation is true:

$$P\left[c_{\alpha/2,n-1} < \frac{(n-1)s^2}{\sigma^2} < c_{1-\alpha/2,n-1}\right] = 1-\alpha. \tag{9.50}$$

We now rearrange this equation to obtain a confidential interval for σ^2. If we take the inverse of all the inside terms, we obtain

$$P\left[\frac{1}{c_{\alpha/2,n-1}} > \frac{\sigma^2}{(n-1)s^2} > \frac{1}{c_{1-\alpha/2,n-1}}\right] = 1-\alpha. \tag{9.51}$$

Note that taking the inverse changes the direction of the inequality signs. Multiplying each term by $(n-1)s^2$ we obtain

$$P\left[\frac{(n-1)s^2}{c_{\alpha/2,n-1}} > \sigma^2 > \frac{(n-1)s^2}{c_{1-\alpha/2,n-1}}\right] = 1-\alpha, \tag{9.52}$$

or equivalently

$$P\left[\frac{(n-1)s^2}{c_{1-\alpha/2,n-1}} < \sigma^2 < \frac{(n-1)s^2}{c_{\alpha/2,n-1}}\right] = 1-\alpha. \tag{9.53}$$

The terms $\frac{(n-1)s^2}{c_{1-\alpha/2,n-1}}$ and $\frac{(n-1)s^2}{c_{\alpha/2,n-1}}$ are the lower and upper $100(1-\alpha)\%$ confidence limits for σ^2, and the interval $(\frac{(n-1)s^2}{c_{1-\alpha/2,n-1}}, \frac{(n-1)s^2}{c_{\alpha/2,n-1}})$ is a $100(1-\alpha)\%$ confidence interval for σ^2 (Mood et al. 1974). The confidence interval for σ^2 is not symmetrical around the value s^2, our estimate of σ^2.

For a 95% confidence interval with $\alpha = 0.05$, the confidence interval formula is

$$\left(\frac{(n-1)s^2}{c_{0.975,n-1}}, \frac{(n-1)s^2}{c_{0.025,n-1}}\right) \tag{9.54}$$

To find $c_{0.025,n-1}$, we look across the top row of Table C and find the column corresponding to $p = 0.025$, then look for the row corresponding to $n-1$ degrees of fredom. To find $c_{0.975,n-1}$, we use the column corresponding to $p = 0.975$, again looking for the row with $n-1$ degrees of freedom.

For a 99% confidence interval with $\alpha = 0.01$, the confidence interval formula is

$$\left(\frac{(n-1)s^2}{c_{0.995,n-1}}, \frac{(n-1)s^2}{c_{0.005,n-1}}\right) \tag{9.55}$$

To find $c_{0.005,n-1}$, we use the column corresponding to $p = 0.005$, while the column for $c_{0.995,n-1}$ corresponds to $p = 0.995$. We again use the entries corresponding to $n-1$ degrees of freedom.

We can also obtain a confidence interval for $\sigma = \sqrt{\sigma^2}$ by taking the square root of the above confidence limits. In particular, a confidence interval for σ would be $(\sqrt{\frac{(n-1)s^2}{c_{1-\alpha/2,n-1}}}, \sqrt{\frac{(n-1)s^2}{c_{\alpha/2,n-1}}})$.

Confidence interval for σ^2 and σ - sample calculation

Recall the elytra data set, for which $\bar{Y} = 5.150$ and $s^2 = 0.109$ for $n = 10$. Calculate a 95% and 99% confidence interval for σ^2 and then σ.

The formula for a 95% confidence interval is

$$\left(\frac{(n-1)s^2}{c_{0.975,n-1}}, \frac{(n-1)s^2}{c_{0.025,n-1}}\right) \tag{9.56}$$

For $n = 10$, we have $df = n - 1 = 10 - 1 = 9$.

For a 95% confidence interval, with $\alpha = 0.05$, we find from Table C that $c_{0.025,n-1} = c_{0.025,9} = 2.700$, and $c_{0.975,n-1} = c_{0.975,9} = 19.023$. Substituting $n = 10, s^2 = 0.110, c_{0.025,9} = 2.700$ and $c_{0.975,9} = 19.023$ in the above formula, we obtain

$$\left(\frac{(10-1)0.109}{19.023}, \frac{(10-1)0.109}{2.700}\right) \tag{9.57}$$

or

$$(0.052, 0.363). \tag{9.58}$$

So, the 95% confidence interval for σ^2 is $(0.052, 0.363)$. To obtain a 95% confidence interval for σ we simply take the square root of these values, or $(\sqrt{0.052}, \sqrt{0.363}$, to obtain $(0.228, 0.603)$.

For a 99% confidence interval, the formula is

$$\left(\frac{(n-1)s^2}{c_{0.995,n-1}}, \frac{(n-1)s^2}{c_{0.005,n-1}}\right) \tag{9.59}$$

We use Table C to find $c_{0.005,n-1} = c_{0.005,9} = 1.735$, and $c_{0.995,n-1} = c_{0.995,9} = 23.589$. Substituting these values in the above formula, we obtain

$$\left(\frac{(10-1)0.109}{23.589}, \frac{(10-1)0.109}{1.735}\right) \tag{9.60}$$

or

$$(0.042, 0.565). \tag{9.61}$$

The 99% confidence interval of σ^2 is therefore $(0.042, 0.565)$. To obtain a 99% confidence interval for σ, we take the square root and obtain $(0.205, 0.752)$. Note that the 99% intervals are wider than the corresponding 95% ones.

9.2.4 Confidence intervals - SAS demo

These same calculations can be readily accomplished using `proc univariate` in SAS (SAS Institute Inc. 2016). We obtain 95% confidence intervals by including the option `cibasic` in the `proc univariate` line of the program.

99% confidence intervals may be obtained by specifying `alpha=0.01` in the `proc univariate` line. See SAS program and Fig. 9.3 - 9.6 below. Similar to our earlier calculations, the 95% confidence interval was $(4.913, 5.387)$ for μ, $(0.052, 0.365)$ for σ^2, and $(0.228, 0.604)$ for σ. The 99% confidence intervals can be found further in the output.

9.2.5 Confidence interval size

Confidence intervals are a method of characterizing the precision of parameter estimates, with narrower intervals generally indicating a population parameter like μ is better estimated. How then can an investigator reduce the size of these confidence intervals? The simplest way is to increase the sample size n on which the estimate is based. This reduces the size of confidence intervals for μ because it reduces the magnitude of the quantity $c_{\alpha,n-1}s/\sqrt{n}$, which determines the width of the interval (see Eq. 9.26). Most of this effect is through the $\sqrt{n}$ term here, but $c_{\alpha,n-1}$ also becomes smaller for larger n. Increasing the sample size n also reduces the size of the confidence intervals for σ^2 and σ, although the mechanism is more complex in this case.

SAS Program

```
* Confidence_intervals.sas;
title 'Confidence intervals for elytra data';
data elytra;
    input length;
    datalines;
5.0
5.1
5.2
5.9
4.8
5.5
4.8
5.1
5.0
5.1
;
run;
* Print data set;
proc print data=elytra;
run;
* Generate 95% confidence intervals and plots;
title2 "95% confidence intervals";
proc univariate cibasic data=elytra;
    var length;
    histogram length / vscale=count normal;
    qqplot length / normal;
run;
* Generate 99% confidence intervals;
title2 "99% confidence intervals";
proc univariate cibasic alpha = 0.01 data=elytra;
    var length;
run;
quit;
```

Confidence intervals for elytra data

Obs	length
1	5.0
2	5.1
3	5.2
4	5.9
5	4.8
6	5.5
7	4.8
8	5.1
9	5.0
10	5.1

Figure 9.3: `confidence_intervals.sas - proc print`

Confidence intervals for elytra data
95% confidence intervals

The UNIVARIATE Procedure
Variable: length

Moments			
N	10	Sum Weights	10
Mean	5.15	Sum Observations	51.5
Std Deviation	0.33082389	Variance	0.10944444
Skewness	1.42698649	Kurtosis	2.26518149
Uncorrected SS	266.21	Corrected SS	0.985
Coeff Variation	6.4237648	Std Error Mean	0.1046157

Basic Statistical Measures			
Location		Variability	
Mean	5.150000	Std Deviation	0.33082
Median	5.100000	Variance	0.10944
Mode	5.100000	Range	1.10000
		Interquartile Range	0.20000

Basic Confidence Limits Assuming Normality			
Parameter	Estimate	95% Confidence Limits	
Mean	5.15000	4.91334	5.38666
Std Deviation	0.33082	0.22755	0.60396
Variance	0.10944	0.05178	0.36476

Figure 9.4: `confidence_intervals.sas - proc univariate`

Confidence intervals for elytra data
99% confidence intervals

The UNIVARIATE Procedure
Variable: length

Moments			
N	10	Sum Weights	10
Mean	5.15	Sum Observations	51.5
Std Deviation	0.33082389	Variance	0.10944444
Skewness	1.42698649	Kurtosis	2.26518149
Uncorrected SS	266.21	Corrected SS	0.985
Coeff Variation	6.4237648	Std Error Mean	0.1046157

Basic Statistical Measures			
Location		Variability	
Mean	5.150000	Std Deviation	0.33082
Median	5.100000	Variance	0.10944
Mode	5.100000	Range	1.10000
		Interquartile Range	0.20000

Basic Confidence Limits Assuming Normality			
Parameter	Estimate	99% Confidence Limits	
Mean	5.15000	4.81002	5.48998
Std Deviation	0.33082	0.20434	0.75349
Variance	0.10944	0.04176	0.56775

Figure 9.5: `confidence_intervals.sas - proc univariate`

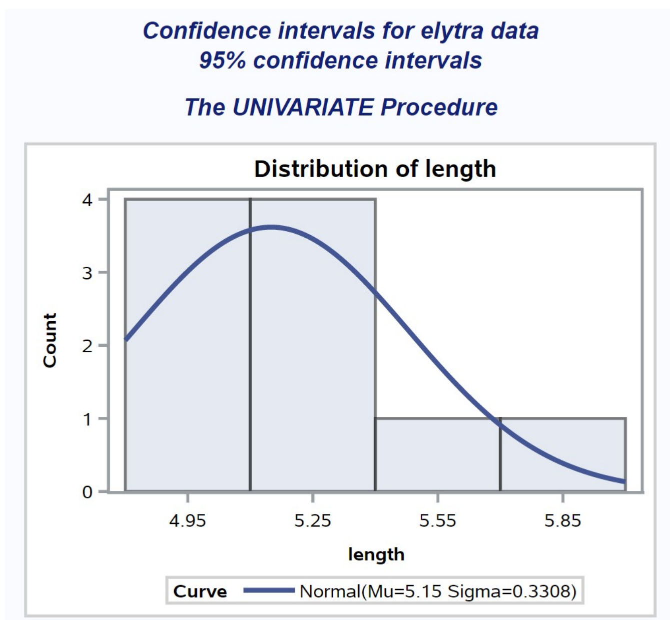

Figure 9.6: `confidence_intervals.sas - proc univariate`

9.3 References

Mood, A. M., Graybill, F. A. & Boes, D. C. (1974) *Introduction to the Theory of Statistics.* McGraw-Hill, Inc., New York, NY.

SAS Institute Inc. (2016) *Base SAS 9.4 Procedures Guide, Sixth Edition.* SAS Institute Inc., Cary, NC.

9.4 Problems

1. Ten adult female *Daphnia ambigua* (Lei and Armitage 1980) were cultured under laboratory conditions, and their longevity (days) determined. The following data were obtained.

   ```
   28 4 22 21 17 21 22 26 15 19
   ```

   (a) Find $\bar{Y}$, s^2, and s for these data, then calculate a 95% confidence interval for μ, σ^2 and then σ. Show all your calculations.

   (b) Find a 99% confidence interval for μ, σ^2 and then σ. Show your calculations.

   (c) Use SAS to find the same confidence intervals as in parts a and b. List the confidence intervals and test results below. Attach your SAS program(s) and output.

2. A study was conducted to measure the population growth rate of a laboratory culture of nematodes. A hundred nematodes were each added to 8 petri dishes of a new growth media, and the number of offspring counted one generation later. The number of offspring divided by the initial number of organisms (100) provides an estimate of λ, the finite growth rate of the population. It is customary to log-transform the values of λ in such studies, yielding $r = \ln(\lambda)$. The following values of r were obtained:

   ```
   2.1 0.8 1.8 1.9 0.8 1.7 0.5 1.6
   ```

   (a) Find $\bar{Y}$, s^2, and s for these data, then calculate a 95% confidence interval for μ, σ^2 and then σ. Show all your calculations.

   (b) Find a 99% confidence interval for μ, σ^2 and then σ. Show your calculations.

   (c) Use SAS to find the same confidence intervals as in parts a and b. List the confidence intervals and test results below. Attach your SAS program(s) and output.